On some metalogical properties of Visser – Yablo sequences and potential infinity

Michał Tomasz Godziszewski

University of Warsaw

The talk discusses some metalogical properties of Visser–Yablo (VY) sequences introduced independently by A. Visser in [6] and S. Yablo in [7]. The latter gave a by now famous example of a semantic paradox that, according to the author, does not involve self-reference. Recall the paradox arises when one considers the following sequence of sentences:

$$\begin{array}{ll} Y_0 & \text{For any } k > 0,\ Y_k \text{ is false.} \\ Y_1 & \text{For any } k > 1,\ Y_k \text{ is false.} \\ Y_2 & \text{For any } k > 2,\ Y_k \text{ is false.} \\ & \vdots \\ Y_n & \text{For any } k > n,\ Y_k \text{ is false.} \\ & \vdots \end{array}$$

In natural language, VY-sequence, taken together with certain truth principles, leads to a paradox: take any sentence Y_n from the sequence and ask what would happen if it was true. Suppose it is. Then, things are as it says, and for any $j > n$ Y_j is false. In particular Y_{n+1} is false and also for any $j > n + 1$ Y_j is false.

But the second conjunct is exactly what Y_{n+1} states, so it turns that Y_{n+1} is true after all. The assumption that Y_n is true led therefore to a contradiction. So it is false. This means that not all sentences following Y_n are false, and so one of them, say Y_k, is true. But then, we can again obtain a contradiction by repeating for Y_k the same reasoning that we have just given for Y_n. So, whether Y_n is true, or false, a contradiction follows. Hence the paradox.

A fruitful study of the paradox formalized over arithmetic (by the results of G. Priest from [5] VY-sentences provably exist in sufficiently rich formal theories, so we indeed can meaningfully reason about them) has revealed that the reasoning has the following interesting feature. In order to derive the contradiction one needs to use a strong assumption concerning the notion of truth: namely one has to assume "for all n, Y_n if and only if $\ulcorner Y_n \urcorner$ is true." $\forall n\ (Y_n \equiv Tr(Y_n))$. If we wanted to replace this *global disquotation* with an infinity of *local disquotation* instances, contradiction could be obtained only if we used some infinitary inference rule (requiring an infinite number of premises) such as the ω-rule.

Specifically, J. Ketland showed in [2] that the theory PAT^- (Peano Arithmetic in the language extended with a truth predicate Tr and induction scheme restricted to arithmetical formulae) with adjoined local disquotation principles for arithmetical sentences (AD) and VY-sentences (YD), is consistent, yet ω-inconsistent. I will also show a simple proof that the theory $PAT + AD + YD$ (with induction scheme for all formulae of the extended language) is also consistent. During the talk I will diverge a bit to the realm of axiomatic truth theories, reporting on some results of C. Cieślinski on VY-sentences in these formal systems.

So far, the story is rather well-known. What is somewhat less known, is that there is a way of handling the paradox which relies on finitistic assumptions. If the world is finite, there aren't enough things in the world to interpret all sentences from the VY-sequence, and the last interpreted one is vacuously true without any threat of paradox. After all, VY-paradox can be thought of as an infinitary version of the Liar paradox, so perhaps thinking it can be dealt with by tackling the notion of infinity isn't completely insane.

Of course, the finitist owes us a story about how they make sense of arithmetic, and how the whole thing should be studied by formal methods. It so happens that formal tools for this task have already been developed. During the talk I will explain what they are, and I'll use them to study the VY-paradox in the finitistic setting. On this approach, it will turn out that things are as we expected: VY-sentences are all false in potentially infinite domains, despite the fact that the framework is rich enough to incorporate sufficiently strong arithmetic.

Specifically, I will prove that under the logic L_{sl} of sufficiently large finite models, defined by M. Mostowski in [4], for any class $\mathcal{K}$ of finite models, if $\mathcal{K} \models_{sl} YD$, then $\mathcal{K} \models_{sl} \neg Y(n)$ for any $n \in \omega$. I will also show the construction of such a class, basing on the notion of FM-domain, introduced by M. Mostowski in [3].

There is, however, a glitch. I'll argue that the way quantifiers are handled in this finitistic setting results in a somewhat scary arithmetical theory. For instance, in a *potentially infinite* domain it turns out that the sentence "there is a greatest number" comes out true without making "n is the greatest number" true for any n. If your goal, as a finitist, is not to revise current mathematics, but to make sense of it in terms of potential infinity, this approach isn't for you.

It turns out that there is another – motivated by the Kripke semantics for modal logic – approach to potential infinity, which has already been used to obtain standard arithmetic, and to make sense of abstraction principles (in the neologicist sense). I will demonstrate the results from a joint work with R. Urbaniak on how this framework handles VY-paradox.

References

[1] C. Cieślinski, *Yablo sequences in truth theories*, in: K. Lodaya (ed.), **Logic and its applications (Lecture Notes in Computer Science LNCS 7750)**, Springer-Verlag, Berlin Heidelberg, 2013, pp. 127-138.

[2] J. Ketland, *Yablo's paradox and ω-inconsistency*, **Synthese** 145 (2005), 295-302.

[3] M. Mostowski, *On representing concepts in finite models*, **Mathematical Logic Quarterly** 47 (2001), 513-523.

[4] M. Mostowski *Truth in the limit*, **Reports on Mathematical Logic** 51 (2016), pp. 75-89.

[5] G. Priest, *Yablo's paradox*, **Analysis** 57 (1997), 236-242.

[6] A. Visser, *Semantics and the liar paradox*, in: D. Gabbay, F. Guenthner (eds.), **Handbook of Philosophical Logic, vol. IV**, Kluwer Academic Publishers, Dordrecht, 1989, pp. 617-706.

[7] S. Yablo, *Paradox without self-reference*, **Analysis** 52 (1993), 251-252.